Flatness, preorders and general metric spaces

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Abstract

We use a generic notion of flatness in the enriched context to define various completions of metric spaces – enrichments over $[0, \infty]$ – and preorders – enrichments over $\mathbf{2}$. We characterize the weights of colimits commuting in $[0, \infty]$ with the *conical terminal object and cotensors*. These weights can be interpreted in metric terms as peculiar filters, the so-called filters *of type* 1. This generalizes Lawvere's correspondence between minimal Cauchy filters and adjoint modules. We obtain a metric completion based on the filters of type 1 as an instance of the *free cocompletion under a class of weights* defined by Kelly. Another class of flat presheaves is considered both in the metric and the preorder context. The corresponding completion for preorders is the so-called *dcpo completion*.

1 Introduction

Many mathematical objects have been fruitfully described as enriched categories: modules, sheaves [3], [26], [27], fibrations [4] and stalks [21] but also, and more simply, metric spaces and preorders [16]. Amongst the interesting properties inherited from category theory, is a general process of (co)completion which consists roughly in adding freely to a category, colimits of a given kind. It is known that these free cocompletions capture many classical cases of completions, as for instance the sheafification and the completion à la Cauchy of metric spaces. The above completion process for categories have been investigated for long and by various authors ([10], [11], [28], [22]) and a general theorem, due to Kelly, may be found in his book [12]. Kelly's theorem asserts that the free addition of *weighted* colimits yields indeed a universal construction. Also the important notion of *closed class of weights* or *Betti's covering* occurred in [3] and was studied in detail in [2] by Albert and Kelly. In a recent paper [14], these classes are considered again and renamed *saturated*. A theorem by Albert and Kelly asserts that the free cocompletion of a small category A under a saturated class Φ of weights is the full subcategory of presheaves over A generated by objects in Φ. From a practical point of view, this result may yield meaningful internal descriptions of free cocompletions. As mentioned in [14], it is the case that any class of *flat* weights is saturated. Let us underline an important point: the definition of flatness in [14] and used here, relies on classes of *weights*

- not on classes of *diagrams*, like for instance in [1]. A few nice results for this generic notion of flatness are recalled in section 2. The purpose of this paper is to investigate a few categorical cocompletions corresponding to various notions of flatness in the somewhat simple setting of metric spaces – enrichments over $[0, \infty]$ – and preorders – enrichments over $\mathbf{2}$.

The results presented in this paper are the following. We studied the notions of flatness based on the following classes of weights:

- $\mathcal{P}_1$ of weights with domains, either the empty category, or the unit category I (with one point $*$, and $I(*, *) = I$ the unit of $\mathcal{V}$);
- $\mathcal{P}_\aleph$ of weights with domains with less than $\aleph$ objects, for an infinite regular cardinal $\aleph$.

In the context of metric spaces, or equivalently enrichments over the base $[0, \infty]$, $\mathcal{P}_1$-flat presheaves on a small A (seen as a category) correspond to particular filters, on A (seen as a metric space). We named these filters *the filters of type* 1. This generalizes the fact that minimal Cauchy filters on A are in one-to-one correspondence with the left adjoint modules $I \longrightarrow\!\!\!\!\!\circ\;\, A$. As a consequence of this, one may use results from (enriched) category theory to develop a theory of convergence for general metric spaces based on the filters of type 1. For instance we can derive a completion of general metric spaces that we formulate in pure topological/metric terms. This completion may be described as a non-symmetrical version of the completion *à la Cauchy*, the so-called *bi-completion* in [7], but it differs from the latter, even for symmetric spaces. Secondary completions can also be deduced by considering subclasses of filters, namely the filters of type $\aleph$, which are the metric counterparts of the $\mathcal{P}_\aleph$-flat weights. These latter completions coincide with the Cauchy-completion for symmetric spaces. They also have a "domain-theoretical" flavor: in the context of preorders and for the case $\aleph = \omega$, the completion obtained is the well-known "dcpo completion".

The topics of general metric spaces has gained popularity since Lawvere's papers ([16] and [17]), especially amongst Computer Scientists ([8], [6], [9], [25], [18], [6], [15], [23] – recently published in [24]). Surprisingly a few of these works really treat general metric spaces explicitly for what they are: enriched categories of a particular kind. This is the view in [6] where completions are defined by considering *ordinary* colimits in the presheaf categories. Also Vickers' recent published work [24] concerns actually a peculiar class of flat presheaves, namely the flat ones with respect to finite conical weights. It is worth comparing his work with the present one. (Both Steve Vickers and the author of the paper were aware of each others' work.) We consider in this paper classes of *weights* of limits and colimits and in particular the class of weights for *cotensors*. This class is actually crucial to characterize the weights corresponding to filters when the base category is $[0, \infty]$. The approach in this work tries to be "as categorical as possible". The author also hopes that it reveals convincing with respect to non-categorical issues. For instance it helps to develop a theory of convergence

for non-symmetric space. It may also precise the essence of the "similarity" between directed complete partial orders and complete metric spaces: both being after all, the algebras of very similar *KZ-doctrines*.

The paper is organized as follows. Section 2 briefly recalls some material about enriched category theory. It aims at precising the terminology, and presenting most of the categorical concepts and results used in the rest of the paper. Next sections are devoted to applications, namely general metric spaces (the base $\mathcal{V}$ is $[0,\infty]$) in section 3 and preorders ($\mathcal{V}$ is $\mathbf{2}$) in section 4.

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2 Background in Enriched Category Theory

For the background knowledge about enriched categories, we refer the reader to Kelly's book [12], augmented by Betti's [3], the Albert-Kelly article [2] and the recent paper [14].

We consider in this section, a symmetric monoidal complete and cocomplete closed $\mathcal{V}$. As usual, weights are presheaves over *small* categories. Remember that given a family Φ of weights, and a small category A, the *closure* $\Phi(A)$ *of* A *under* Φ*-colimits in* $[A^{op},\mathcal{V}]$, is defined as the smallest full replete subcategory of the presheaf category $[A^{op},\mathcal{V}]$, containing the representables and closed under the formation of Φ-colimits in $[A^{op},\mathcal{V}]$. The latter means that for any weights $F : K^{op} \to \mathcal{V} \in \Phi$, and any functor $G : K \to [A^{op},\mathcal{V}]$ taking its values in $\Phi(A)$, $F * G$ is again in $\Phi(A)$. Remember also that for a small A, $\Phi(A)$ is generally not small.

Theorem 2.1 *([12] Theorem 5.35) For any family of weights* Φ*, and for any small* A*, the closure* $\Phi(A)$ *of* A *in* $[A^{op},\mathcal{V}]$ *under* Φ*-colimits constitutes the* free Φ-cocompletion *of* A*. Precisely this means that:*

- $\Phi(A)$ *is* Φ*-cocomplete;*
- *For any possibly large* Φ*-cocomplete category* B*, one has an equivalence*

$$(*)\ Lan_K : \mathcal{V}\text{-}Cat(A,B) \cong \Phi\text{-}Cocts(\Phi(A),B)$$

where:

- K *is the full and faithful inclusion* $A \to \Phi(A)$ *sending any* $a \in A$ *to* $A(-,a)$*;*
- Φ*-*$Cocts(\Phi(A),B)$ *is the full subcategory of* $\mathcal{V}$*-*$Cat(\Phi(A),B)$ *generated by the* Φ*-cocontinuous functors;*

- Lan_K *stands for the* left Kan extension functor *and has inverse the functor "$- \circ K$" given by the composition with K.*

Actually the equivalence $(*)$ above lifts to a $\mathcal{V}$-equivalence since the A considered is small.

Definition 2.2 *Given a class Φ of weights, its* saturation *is the largest class Φ^* such that every Φ-complete category is also Φ^*-complete, and every Φ-continuous functor is also Φ^*-continuous.*

Note that in the definition above, one can substitute "Φ-cocomplete" to "Φ-complete" and "Φ-cocontinuous" to "Φ-continuous". A class Φ of weights is called *saturated* if and only if $\Phi = \Phi^*$. For any small $\mathcal{V}$-category A, and any family Φ of weights, we write $\Phi[A]$ for the full subcategory of $[A^{op}, \mathcal{V}]$ with objects in Φ. Note that by Yoneda, any weight F occurs as a colimit weighted by F of representables: $F \cong F * Y$, so that for any class Φ of weights and any small category A, certainly $\Phi[A] \subset \Phi(A)$.

We can now formulate the Albert-Kelly theorem [2].

Theorem 2.3 *For any class Φ of weights, a weight $\phi : A^{op} \to \mathcal{V}$ lies in Φ^* if and only if $\phi \in \Phi(A)$. Which is to say that $\Phi^*[A] = \Phi(A)$ for all class Φ and all small A.*

Corollary 2.4 *A class Φ is saturated if and only if $\Phi[A] = \Phi(A)$ for all small category A.*

It is our view that the latter characterization of saturated classes can reveal useful to obtain internal descriptions of free cocompletions. As shown in [14], an important example of saturated classes is given by classes of *flat* weights that we introduce now.

Given a class Φ of weights, a weight $F : A^{op} \to \mathcal{V}$ is said Φ-*flat* when its left Kan extension along Y, $- * F : [A, \mathcal{V}] \to \mathcal{V}$ preserves all Φ-limits. This is also equivalent to say that the "weighting by F" functor $F * - : [A, \mathcal{V}] \to \mathcal{V}$ – that is isomorphic to $- * F$ – is Φ-continuous. As precisely stated in [14], F is Φ-flat when Φ-*limits commute with F-colimits in the base $\mathcal{V}$*. Letting Φ^+ denote the family of Φ-flat weights, it happens that the class Φ^+ is saturated. Thus by the Albert-Kelly theorem one has:

Theorem 2.5 *For any family Φ of weights and any small category A, $\Phi^+[A]$ is the free Φ^+-cocompletion $\Phi^+(A)$ of A.*

We wish to mention here another nice consequence of the fact that classes of the form Φ^+ are saturated. This is obtained by combining Theorem 2.5 above with the characterization (due to Kelly) of the embeddings of the form $A \to \Phi(A)$ (see again [14], Propositions 4.2 and 4.3). For a class Φ and a category A, we write A_Φ for its full subcategory defined by the a such that $A(a, -) : A \to \mathcal{V}$ preserves Φ-colimits.

Theorem 2.6 *For any class Φ of weights, if Ψ stands for Φ^+, then for a category A, the following are equivalent:*
- A is Ψ-cocomplete and has a full small category $B \subset A_\Psi$ such that any $a \in A$ is a Ψ-colimit of a diagram in B;
- there exists a small B such that A is equivalent to the full subcategory of $[B^{op}, \mathcal{V}]$ defined by Φ-flat weights.

The previous theorem can be seen as a generalization of a well-known result regarding accessible categories. The work [14] contains a few examples of notions of flatness. Let us recall in particular that for the empty class 0 of weights and the class $\mathcal{P}$ of all weights, one has $0^+ = \mathcal{P}$ whereas $\mathcal{P}^+$ is the class denoted $\mathcal{Q}$ of *small projectives*, which are also the weights $A^{op} \to \mathcal{V}$ defining left adjoint modules $I \longrightarrow\!\!\circ\!\!\to A$, and also the *weights of absolute colimits* ([20]). So actually for any small category A, $\mathcal{P}^+(A) = \mathcal{Q}(A)$ is the Lawvere-Cauchy-completion A.

In this work, we shall investigate a few classes of colimits commuting with certain given limits in $[0, \infty]$ and in **2**. Precisely we shall study the notions of flatness associated to the the following classes of weights (that we morally think as weights of limits):

- $\mathcal{P}_1$: the class of weights of the form $F : K \to \mathcal{V}$ where K is the empty $\mathcal{V}$-category or $K = I$ (the unit category with one point $*$ and $I(*, *) = I$);
- $\mathcal{P}_\aleph$, for any infite regular cardinal $\aleph$: the class of weights $F : K \to \mathcal{V}$ such that and $\sharp Obj(K) < \aleph$.

We need now to consider "special" weighted limits, namely *cotensors* and *conical limits*. We skip their definitions, referring the reader to the section 3 of Kelly's book [12]. We recall anyhow a few facts regarding *conical limits* to avoid a too common confusion with the mere ordinary limits. Conical limits (and dually colimits) are defined as representatives of particular presheaves and their defining isomorphisms are $\mathcal{V}$*-natural*. Though any conical limit has an underlying ordinary limit, the converse does not hold. There is an alternative characterization of canonical limits: they are the ordinary limits which are preserved by ordinary functors underlying representables (see [12] p.95). This point is crucial for instance in the proof of Theorem [12] (3.73) and thus in the proof below of Proposition 2.9.

According to Kelly's terminology, $\mathcal{P}_1$ is the class of weights for the conical terminal object and cotensors. One has the following inclusions of classes of weights, where $\aleph$ stands for any infinite regular cardinal:

2.7 $\mathcal{P}_1 \subset \mathcal{P}_\omega \subset \mathcal{P}_\aleph \subset \bigcup_\aleph \mathcal{P}_\aleph = \mathcal{P}$

and thus

2.8 $\mathcal{P}_1^+ \supset \mathcal{P}_\omega^+ \supset \mathcal{P}_\aleph^+ \supset \bigcap_\aleph \mathcal{P}_\aleph^+ = \mathcal{Q}$.

For any infinite regular cardinal $\aleph$, we call a limit with weights in $\mathcal{P}_\aleph$, an $\aleph$-*limit*. Thus a *conical* $\aleph$-*limit* is just a conical limit whose underlying ordinary limit has an indexing diagram with less than $\aleph$ objects. A minor adaptation of the proof of theorem [12] (3.73) like in [13] (4.3), shows the following.

Proposition 2.9 *For any infinite regular cardinal $\aleph$, a $\mathcal{V}$-category A is $\mathcal{P}_\aleph$-complete if and only if it has cotensors and all conical $\aleph$-limits. Given a $\mathcal{P}_\aleph$-complete A, a $\mathcal{V}$-functor $P : A \to B$ is $\mathcal{P}_\aleph$-continuous if only if it preserves conical $\aleph$-limits and cotensors.*

PROOF: (Sketch of) It suffices to reuse the argument developed in the sketch of proof of [13] (4.3). Remark that if A has all conical $\aleph$-limits and cotensors, then for any $F : K \to \mathcal{V}$ and $G : K \to A$ with $\sharp Obj(K) < \aleph$, the weighted limit $\{F, G\}$ may be computed as the equalizer in A_0 of the canonical pair

$$\prod_{k\in K} Fk \pitchfork Gk \rightrightarrows \prod_{k,k'\in K} K(k,k') \pitchfork (Fk \pitchfork Gk')\ .$$

Actually all the ordinary limits involved in this equalizer, i.e. the two products and the equalizer itself, have indexing diagrams with less than $\aleph$ objects and thus are conical. Also revisiting the sketched proof of theorem (3.73) in [12], one gets that for any functor $H : A \to B$ preserving conical $\aleph$-limits and cotensors, H will preserve an equalizer as above which image in B is then the limit $\{F, HG\}$. ∎

We finish this quick tour of category theory by mentioning miscellaneous results that will serve later. The following proposition already occurs in [14] but under a more elaborate form.

Proposition 2.10 *For any saturated class Ψ of weights, any weight $F : A \to \mathcal{V}$ in Ψ, and any functor $G : A \to B$ with B small, the left Kan extension of F along G is again in Ψ.*

PROOF: Since $F \cong F * Y$, F is a Ψ-colimit of representables. The image by $Lan_G : [A, \mathcal{V}] \to [B, \mathcal{V}]$ of any representable is again representable (For any $a \in A$, $Lan_G(A(a,-))(b) \cong_b B(G-, b) * A(a,-) \cong_b B(Ga, b)$). Also the left Kan extension functor $Lan_G : [A, \mathcal{V}] \to [B, \mathcal{V}]$ is cocontinuous since it is a left adjoint, so $Lan_G(F)$ is also a Ψ-colimit of representables hence it belongs to Ψ since Ψ is saturated. ∎

Another point is that when the base $\mathcal{V}$ is small – and this is the case for our applications with $\mathcal{V} = \mathbf{2}$ and $\mathcal{V} = [0, \infty]$ – then $\mathcal{V}$ being small-complete is necessarily a preorder (see for instance [5] prop. 2.7.1 p.59). So in the case of a small $\mathcal{V}$, for any small $\mathcal{V}$-category A, the presheaf category $[A, \mathcal{V}]$ remains small and so does $\Phi(A)$ for any family Φ of weights.

Recall also that if a small category A is Φ-cocomplete then it is a retract of $\Phi(A)$ (i.e. the inclusion $A \to \Phi(A)$ is a split monic) but it is generally NOT

isomorphic to A. This situation has been studied (see for instance [3] Theorem p.175, or [2] Proposition 4.5) and it happens that a category A is Φ-cocomplete if and only if the inclusion $A \to \Phi(A)$ is a left adjoint. Nevertheless for some Φ and some Φ-cocomplete A, it happens sometimes that $\Phi(A) \simeq A$. This is the case for all Φ-cocomplete A when Φ is the class $\mathcal{Q}$ of small projective weights. As we shall see now, this also happens for the base $A = \mathcal{V}$ when Φ is class $\mathcal{C}$ of weights of cotensors – i.e. those weights of the form $I \to \mathcal{V}$.

Proposition 2.11 *For any monoidal closed complete and cocomplete $\mathcal{V}$, $\mathcal{V} \simeq \mathcal{C}^+(\mathcal{V})$.*

PROOF: Let us establish first the following result.

Lemma 2.12 *Let M and N be two accessible presheaves $\mathcal{V}^{op} \to \mathcal{V}$, such that N is $\mathcal{C}$-flat, then $\{M, N\} = [M * 1, N * 1]$.*

PROOF: For any $v \in \mathcal{V}$, one has $[M * 1, v] \cong \{M, [-, v]\}$ so that $[M * 1, N * 1] = \{M, [-, N * 1]\}$, the right hand side here is $\{M_-, N_? * [-, ?]\} = \{M, N\}$ since N is $\mathcal{C}$-flat and colimits are pointwise in functor categories. ∎

As a consequence of the previous lemma, the restriction of $- * 1 : [\mathcal{V}^{op}, \mathcal{V}] \to \mathcal{V}$ to $\mathcal{C}^+(\mathcal{V})$ is fully faithful. Since this functor is also essentially surjective on objects, it is part of an equivalence of categories, with inverse the inclusion $\mathcal{V} \to [\mathcal{V}^{op}, \mathcal{V}]$. ∎

3 General metric spaces – The case $\mathcal{V} = [0, \infty]$.

This section applies the categorical results presented in the previous section in the context of general metric spaces. Let us start by recalling briefly a few results that are from [16] or belong to folklore. $[0, \infty]$ stands for the monoidal closed category with:

- objects: nul or positive reals and $+\infty$;
- arrows: the reverse ordering, $x \to y$ if and only if $x \geq y$;
- tensor: the addition (with $+\infty + x = x + +\infty = +\infty$);
- unit: 0.

For any pair x, y of objects in $[0, \infty]$, the exponential object $[x, y]$ is $max\{y - x, 0\}$.

A *small* $[0, \infty]$*-category* A corresponds to a *general metric space*. That is a set of *objects* or *elements*, $Obj(A)$ (most of the time just denoted by A) together with a map $A(-, -) : Obj(A) \times Obj(A) \to [0, \infty]$, called *pseudo-distance*, that satisfies:

- for all $x, y, z \in A$, $A(y, z) + A(x, y) \geq A(x, z)$;
- for all $x \in A$, $0 \geq A(x, x)$.

A $[0, \infty]$-functor $F : A \to B$ corresponds to a *non-expansive* map $F : Obj(A) \to Obj(B)$, i.e. for all $x, y \in A$, $A(x, y) \geq B(F(x), F(y))$. A $[0, \infty]$-natural transformation $F \Rightarrow G : A \to B$ corresponds to the fact that for all $x \in A$, $0 \geq B(F(x), G(x))$. A $[0, \infty]$-module $M : I \longrightarrow\!\!\circ\!\!\to A$ – or left module on A – is a map $Obj(A) \to [0, \infty]$ such that for all $x, y \in A$, $M(y) + A(x, y) \geq M(x)$. Dually a $[0, \infty]$-module $N : A \longrightarrow\!\!\circ\!\!\to I$ – or right module on A – is a map $Obj(A) \to [0, \infty]$ such that for all $x, y \in A$, $A(x, y) + N(x) \geq N(y)$. For any general metric spaces A and B, the set of non expansive maps from A to B becomes a general metric space $[A, B]$ with distance $[A, B](f, g) = \bigvee_{x \in A} B(f(x), g(x))$. In particular the presheaf category $[A^{op}, [0, \infty]]$ has hom-sets given by $[A^{op}, [0, \infty]](M, N) = \bigvee_{x \in A}[M(x), N(x)]$. Its underlying category is a partial order with arrows given by the pointwise reverse ordering: $M \Rightarrow N$ if and only if $\forall x \in A$, $M(x) \geq N(x)$. The composition of left and right modules is as follows. Given $I \xrightarrow{M}\!\!\circ\, A \xrightarrow{N}\!\!\circ\, I$, the composite $N * M$ is $\bigwedge_{x \in A} M(x) + N(x)$. For such M and N, M is left adjoint to N if and only if:

- (1) $0 \geq N * M$;
- (2) for all $x, y \in A$, $N(y) + M(x) \geq A(x, y)$.

For the rest of this section, A denotes an arbitrary general metric space that we freely see as a small category.

Lawvere observed that Cauchy sequences on the space A correspond to (left) adjoint modules $I \longrightarrow\!\!\circ\!\!\to A$. Actually there is a bijection between left adjoint modules on the small $[0, \infty]$-category A and minimal Cauchy filters on the space A. From this observation mainly, one gets that the full subcategory of $[A^{op}, [0, \infty]]$ generated by left adjoint modules $I \longrightarrow\!\!\circ\!\!\to A$, corresponds to the completion à la Cauchy of the space A if A is symmetric or in general to its bi-completion (see for instance [7] and [8] or [19]).

Definition 3.1 *A filter $\mathcal{F}$ on A is* Cauchy *if and only if for any $\epsilon > 0$, there exists an $f \in \mathcal{F}$ such that for any elements x, y of f, $A(x, y) \leq \epsilon$ or equivalently when:*

$$\bigwedge_{f \in \mathcal{F}} \bigvee_{x, y \in f} A(x, y) = 0.$$

Definition 3.2 *For any left adjoint module M on A, with right adjoint $\tilde{M}$, $\Gamma^s(M)$ stands for the subset of the powerset $\wp(A)$ of A: $\{\Gamma^s(M)(\epsilon) \mid \epsilon \in]0, +\infty]\}$, where $\Gamma^s(M)(\epsilon)$ denotes the set $\{x \in A \mid M(x) + \tilde{M}(x) \leq \epsilon\}$.*

For any left adjoint module M on A, $\Gamma^s(M)$ is a Cauchy basis. The filter that it generates, that we denote $\mathcal{F}^s(M)$, is a minimal Cauchy filter. The map $M \mapsto \mathcal{F}^s(M)$ defines a bijection between left adjoint modules $I \longrightarrow\!\!\circ\!\!\to A$ and

minimal Cauchy filters on A. One may check the following points (proved for instance in [19]). To any Cauchy filter $\mathcal{F}$, one may associate a left adjoint module $M^l(\mathcal{F})$ defined by

$$x \mapsto \bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(x,y) = \bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(x,y).$$

$M^l(\mathcal{F})$ has right adjoint $M^r(\mathcal{F})$ given by the map

$$x \mapsto \bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(y,x) = \bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(y,x).$$

For any left adjoint module M on A, $M^l(\mathcal{F}^s(M)) = M$ and for any Cauchy filter $\mathcal{F}$ on A, $\mathcal{F}^s(M^l(\mathcal{F}))$ is the only minimal Cauchy filter contained in $\mathcal{F}$.

3.1 Modules and Filters

Since the (Lawvere-)Cauchy-completion of A is up to equivalence the full subcategory of flat presheaves $\mathcal{P}^+(A)$, we wondered whether the previous correspondence between left adjoint modules and Cauchy filters could be extended to larger classes of Φ-flat modules and filters. We are going to show in 3.14 that this is the case for $\Phi = \mathcal{P}_1$ as we exhibit a $[0,\infty]$-category structure $Fil_1(A)$ on the set of the so-called *filters of type* 1 on A, which is equivalent to the category $\mathcal{P}_1^+(A)$.

We need to recall a few technical points before giving simple characterizations of the $\mathcal{P}_1$ and $\mathcal{P}_\aleph$-flat presheaves. For the assertions 3.3, 3.4 and 3.5 below, $\mathcal{V}$ denotes a complete and cocomplete monoidal closed category. Remember that cotensors are defined pointwise in functor categories. In particular:

3.3 *For any category C, the presheaf $\mathcal{V}$-category $[C,\mathcal{V}]$ is cotensored and for any presheaf N, $v \pitchfork N$ is the composite* $C \xrightarrow{N} \mathcal{V} \xrightarrow{[v,-]} \mathcal{V}$.

Also for functors between *cocomplete* categories, the preservation of conical colimits amounts to the preservation of ordinary colimits. Precisely one may check:

3.4 *Given a $\mathcal{V}$-functor $T : C \to D$ with underlying ordinary functor $T_0 : C_0 \to D_0$ and an ordinary functor $P : J \to C_0$ with J small, if the conical limits of P and of T_0P exist and T_0 preserves the ordinary limit of P, then T preserves the conical limit of P.*

Eventually the preservation of limits/colimits is simple in the case $\mathcal{V} = [0,\infty]$, according to the following observation.

3.5 *If the base category $\mathcal{V}_0$ is a preorder, then given a weight $F : B^{op} \to \mathcal{V}$, a functor $G : B \to C$ such that $F * G$ exists and a functor $H : C \to D$, H preserves $F * G$ if and only if $F * (GH)$ exists and $H(F * G) \cong F * (GH)$.*

According to the three previous points one gets the following results.

3.6 *Let $M : I \longrightarrow\!\!\!\!\circ\!\!\rightarrow A$ be a left module.*

- $- * M : [A, [0,\infty]] \to [0,\infty]$ *preserves the unique conical limit with weight with empty domain if and only if the underlying ordinary functor preserves the terminal object i.e.* $0 * M = 0$ *if and only if*

$$(1) \quad \bigwedge_{x \in A} M(x) = 0.$$

- *For an infinite regular cardinal $\aleph$, $- * M$ preserves the conical $\aleph$-limits if and only if*
  (2) *for any family of right modules $N_i : A \longrightarrow\!\!\!\!\circ\!\!\rightarrow I$, where $i \in I$ and $\sharp I < \aleph$,*

$$\bigwedge_{x \in A} (M(x) + \bigvee_{i \in I} N_i(x)) = \bigvee_{i \in I} (\bigwedge_{x \in A} M(x) + N_i(x));$$

- $- * M$ *preserves cotensors if and only if*
  (3) *for any $v \in [0,\infty]$ and any right module $N : A \longrightarrow\!\!\!\!\circ\!\!\rightarrow I$,*

$$\bigwedge_{x \in A} (M(x) + [v, N(x)]) = [v, \bigwedge_{x \in A} (M(x) + N(x))].$$

So $\mathcal{P}_1$-flat modules are the modules satisfying (1) and (3) above, whereas for an infinite regular cardinal $\aleph$, the $\mathcal{P}_\aleph$-flat modules are those satisfying (2) and (3).

We introduce now the filters that will occur as the metric counterparts of the $\mathcal{P}_1$-flat modules.

Definition 3.7 *Given a filter $\mathcal{F}$ on A and a map $f : Obj(A) \to Obj([0,\infty])$, $lim^+_{x \in \mathcal{F}} f(x)$, or simply $lim^+_{\mathcal{F}} f$, denotes $\bigwedge_{f \in \mathcal{F}} \bigvee_{x \in f} f(x)$ and $lim^-_{x \in \mathcal{F}} f(x)$, or $lim^-_{\mathcal{F}} f$, denotes $\bigvee_{f \in \mathcal{F}} \bigwedge_{x \in f} f(x)$.*

Definition 3.8 *A filter $\mathcal{F}$ on A has* type 1 *if and only if*

$$lim^+_{x \in \mathcal{F}} lim^-_{y \in \mathcal{F}} A(x, y) = 0.$$

Now compare the previous definition and 3.1. Definition 3.8 is a generalization to non-symmetric spaces of the fact that the diameter of the elements of the filter $\mathcal{F}$ may be chosen arbitrary small.

Remark 3.9 *Any Cauchy filter has type* 1.

From the correspondence between Cauchy filters and left adjoint modules, we know two operators associating filters to modules.

Definition 3.10 *Given any filter $\mathcal{F}$ on A, we define the following* $[0,\infty]$*-valued maps on objects of A:*

$$\begin{array}{l} M^-(\mathcal{F}) : x \mapsto lim^-_{\mathcal{F}} A(x,-) = \bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(x,y), \\ M^+(\mathcal{F}) : x \mapsto lim^+_{\mathcal{F}} A(x,-) = \bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(x,y). \end{array}$$

For any filter $\mathcal{F}$ on A, one has $M^-(\mathcal{F}) \leq M^+(\mathcal{F})$ where the order is pointwise, and if $\mathcal{F}$ is Cauchy then $M^-(\mathcal{F}) = M^+(\mathcal{F})$.

Remark 3.11 *A filter $\mathcal{F}$ on A has* type 1 *if and only if*

$$lim^+_{\mathcal{F}} M^-(\mathcal{F}) = 0.$$

3.12 *Given any filter $\mathcal{F}$ on A, the map $x \mapsto M^-(\mathcal{F})(x)$ defines a module $I \longrightarrow\!\!\circ\!\!\to A$.*

PROOF: For all $x, y \in A$,

$$\begin{array}{rl} M^-(\mathcal{F})(x) + A(y,x) & = (\bigvee_{f\in\mathcal{F}} \bigwedge_{z\in f} A(x,z)) + A(y,x) \\ & \geq \bigvee_{f\in\mathcal{F}} ((\bigwedge_{z\in f} A(x,z)) + A(y,x)) \\ & = \bigvee_{f\in\mathcal{F}} \bigwedge_{z\in f} (A(x,z) + A(y,x)) \\ & \geq \bigvee_{f\in\mathcal{F}} \bigwedge_{z\in f} A(y,z) \\ & = M^-(\mathcal{F})(y). \end{array}$$

∎

Conversely assigning a filter to a module should be a simple matter.

Definition 3.13 *For any module M, $\Gamma(M)$ denotes the subset $\{\Gamma(M)(\epsilon) \mid \epsilon \in]0,+\infty]\}$ of $\wp(A)$, where $\Gamma(M)(\epsilon)$ is the set $\{x \in A \mid M(x) \leq \epsilon\}$. Also $\mathcal{F}(M)$ denotes the upper closure of $\Gamma(M)$ in $\wp\wp(A)$ ordered by inclusion.*

We can now state our main result:

Theorem 3.14 *The set of filters of type 1 on A may be given a general metric space structure $Fil_1(A)$ that is equivalent to $\mathcal{P}^+_1(A)$. The "distance" on $Fil_1(A)$ is defined by the map*

$$(\mathcal{F}_1, \mathcal{F}_2) \mapsto lim^+_{x\in\mathcal{F}_1} lim^-_{y\in\mathcal{F}_2} A(x,y).$$

The functors of this equivalence are defined by the maps on objects $M \mapsto \mathcal{F}(M)$ and $\mathcal{F} \mapsto M^-(\mathcal{F})$.

We prove now a succession of results that constitutes the proof of Theorem 3.14.

Proposition 3.15 *For any module $N :\ A \longrightarrow\!\!\circ\!\!\to I$, and any filter $\mathcal{F}$ on A,*

$$N * M^-(\mathcal{F}) \geq lim^-_{\mathcal{F}} N.$$

Moreover if $\mathcal{F}$ is of type 1 then the previous inequality becomes an equality.

PROOF: For any module N and any filter $\mathcal{F}$ as above,

$$\begin{aligned}N * M^-(\mathcal{F}) &= \bigwedge_{x\in A}(M^-(\mathcal{F})(x) + N(x))\\ &= \bigwedge_{x\in A}((\bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(x,y)) + N(x))\\ &\geq \bigwedge_{x\in A} \bigvee_{f\in\mathcal{F}}((\bigwedge_{y\in f} A(x,y)) + N(x))\\ &= \bigwedge_{x\in A} \bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f}(A(x,y) + N(x))\\ &\geq \bigvee_{f\in\mathcal{F}}(\bigwedge_{y\in f} N(y)).\end{aligned}$$

Let us suppose moreover that $\mathcal{F}$ has type 1. Let $\epsilon > 0$. One may choose $f_\epsilon \in \mathcal{F}$ such that when $x \in f_\epsilon$, $M^-(\mathcal{F})(x) \leq \epsilon$. Thus

$$\begin{aligned}N * M^-(\mathcal{F}) &= \bigwedge_{x\in A}(M^-(\mathcal{F})(x) + N(x))\\ &\leq M^-(\mathcal{F})(x) + N(x), \text{ for any } x \in f_\epsilon\\ &\leq \epsilon + N(x), \text{ for any } x \in f_\epsilon.\end{aligned}$$

Thus

$$\begin{aligned}N * M^-(\mathcal{F}) &\leq \bigwedge_{x\in f_\epsilon}(\epsilon + N(x))\\ &= \epsilon + \bigwedge_{x\in f_\epsilon} N(x)\\ &\leq \epsilon + \bigvee_{f\in\mathcal{F}} \bigwedge_{x\in f} N(x).\end{aligned}$$

∎

3.16 *If $\mathcal{F}$ is a filter of type 1 on A then $- * M^-(\mathcal{F})$ preserves cotensors.*

The proof of this fact relies on a very peculiar property of the base $[0,\infty]$, namely that *cotensors commute in $[0,\infty]$ with conical colimits of non-empty diagrams.* This is to say:

3.17 *For any v in $[0,\infty]$ and any non empty family $(a_i)_{i\in I}$ in $[0,\infty]$,*

$$[v, \bigwedge_{i\in I} a_i] = \bigwedge_{i\in I}[v, a_i].$$

We just need to prove $[v, \bigwedge_{i\in I} a_i] \geq \bigwedge_{i\in I}[v, a_i]$. Let us fix $\epsilon > 0$. Since I is not empty, there exists $j \in I$ such that $\epsilon + \bigwedge_{i\in I} a_i \geq a_j$. Also $[v, \bigwedge_{i\in I} a_i] \geq [v, \bigwedge_{i\in I} a_i]$, so $v+[v, \bigwedge_{i\in I} a_i] \geq \bigwedge_{i\in I} a_i$. For a j as above, $\epsilon+v+[v, \bigwedge_{i\in I} a_i] \geq a_j$ and $\epsilon + [v, \bigwedge_{i\in I} a_i] \geq [v, a_j] \geq \bigwedge_{i\in I}[v, a_i]$. ∎

We can now prove 3.16.

PROOF: Given $v \in [0,\infty]$ and $N : A \longrightarrow\!\!\!\!\!\circ\;\; I$ we have to show $(v \pitchfork N) * M^-(\mathcal{F}) = [v, N*M^-(\mathcal{F})]$. According to 3.15, $(v \pitchfork N)*M^-(\mathcal{F}) = \bigvee_{f\in\mathcal{F}} \bigwedge_{x\in f}[v, N(x)]$ and $[v, N * M^-(\mathcal{F})] = [v, \bigvee_{f\in\mathcal{F}} \bigwedge_{x\in f} N(x)] = \bigvee_{f\in\mathcal{F}}[v, \bigwedge_{x\in f} N(x)]$. Since all the $f \in \mathcal{F}$ are non empty, the result follows then from 3.17. ∎

3.18 *If $\mathcal{F}$ is a filter of type 1 on A then $- * M^-(\mathcal{F})$ preserves the terminal object.*

PROOF: We have to show that $\bigwedge_{x\in A} M^-(\mathcal{F})(x) = 0$. For any $\epsilon > 0$, since $lim^+_{\mathcal{F}} M^-(\mathcal{F}) = 0$, one may find an $f \in \mathcal{F}$ such that for any $x \in f$, $M^-(\mathcal{F})(x) \leq \epsilon$. Since that f is not empty, $\bigwedge_{x\in A} M^-(\mathcal{F})(x) \leq \epsilon$. ∎

According to 3.16 and 3.18, for any filter $\mathcal{F}$ of type 1, the module $M^-(\mathcal{F})$ is $\mathcal{P}_1$-flat. Therefore we shall define a general metric structure $Fil_1(A)$ on the set of filters of type 1 by letting

$$Fil_1(A)(\mathcal{F}_1, \mathcal{F}_2) = [A^{op}, [0, \infty]](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)).$$

(We shall simplify this distance later on.) A consequence of this definition is that the map $\mathcal{F} \mapsto M^-(\mathcal{F})$ defines a fully faithful functor $Fil_1(A) \to \mathcal{P}_1^+(A)$.

3.19 *A filter $\mathcal{F}$ on A has type 1 if and only if $\mathcal{F} \supset \mathcal{F} \circ M^-(\mathcal{F})$.*

PROOF: One has the successive equivalences.

$\mathcal{F}$ has type 1
if and only if
$\bigwedge_{f\in\mathcal{F}} \bigvee_{x\in f} M^-(\mathcal{F})(x) = 0$
if and only if
for all $\epsilon > 0$, there exists $f \in \mathcal{F}$ such that for all $x \in f$, $M^-(\mathcal{F})(x) \leq \epsilon$,
if and only if
for all $\epsilon > 0$, there exists $f \in \mathcal{F}$ such that $f \subset \Gamma(M^-(\mathcal{F}))(\epsilon)$
if and only if
$\mathcal{F} \supset \mathcal{F} \circ M^-(\mathcal{F})$.

∎

3.20 *For any module $M : I \longrightarrow\!\!\!\!\!\circ\!\!\!\!\!\longrightarrow A$ if $- * M : [A, [0, \infty]] \to [0, \infty]$ preserves the terminal object (i.e. $\bigwedge_{x\in A} M(x) = 0$) then $\mathcal{F}(M)$ is a filter on A with basis the family $\Gamma(M)$.*

PROOF: Let us see first that the set of subsets of the form $\Gamma(M)(\epsilon)$ for $\epsilon > 0$, is a filter basis on A. Since $\bigwedge_{x\in A} M(x) = 0$, for any $\epsilon > 0$ there is one x with $M(x) \leq \epsilon$, i.e. $\Gamma(M)(\epsilon) \neq \emptyset$. That $\Gamma(M)$ is a cofiltered subset of $\wp(A)$ ordered by inclusion is trivial. ∎

3.21 *If M is a left module on A then for all x,*

$$M(x) \leq \bigvee_{\epsilon>0} \bigwedge_{y|M(y)\leq\epsilon} A(x, y).$$

PROOF: Let $x \in A$. For all $y \in A$, $M(x) \leq M(y) + A(x, y)$ thus for all $y \in A$, such that $M(y) \leq \alpha$, $M(x) \leq A(x, y) + \alpha$ and $M(x) \leq \bigwedge_{y|M(y)\leq\alpha} A(x, y) + \alpha$. Consider $\epsilon > 0$. The map $\alpha \mapsto \bigwedge_{y|M(y)\leq\alpha} A(x, y)$ reverses the order so $\bigwedge_{y|M(y)\leq\epsilon} A(x, y) = \bigvee_{\alpha\geq\epsilon} \bigwedge_{y|M(y)\leq\alpha} A(x, y)$ and

$$(*)\ M(x) \leq \bigvee_{\alpha \geq \epsilon} \bigwedge_{y | M(y) \leq \alpha} A(x,y) + \epsilon.$$

Also for any $\alpha \leq \epsilon$,

$$\begin{aligned} M(x) &\leq (\bigwedge_{y|M(y)\leq\alpha} A(x,y)) + \alpha \\ &\leq (\bigwedge_{y|M(y)\leq\alpha} A(x,y)) + \epsilon \end{aligned}$$

and thus

$$(**)\ M(x) \leq \bigvee_{\alpha \leq \epsilon} \bigwedge_{y|M(y)\leq\alpha} A(x,y) + \epsilon.$$

$(*)$ and $(**)$ give $M(x) \leq \bigvee_{\alpha > 0} \bigwedge_{y|M(y)\leq\alpha} A(x,y) + \epsilon$. ∎

3.22 *If the module $M : I \longrightarrow\!\!\!\!\circ\; A$ is such that $- * M$ preserves cotensors then $M(x) \geq \bigvee_{\alpha>0} \bigwedge_{M(y)\leq\alpha} A(x,y)$.*

PROOF: We show that for any $\epsilon > 0$ and any x with $M(x) < \epsilon$ and any $\alpha > 0$, there is a y such $M(y) \leq \alpha$ and $A(x,y) \leq \epsilon$. Consider $\epsilon > 0$ and x with $M(x) < \epsilon$. Then

$$\begin{aligned} 0 &= [M(x), M(x)] \\ &= [M(x), A(x,-) * M] \\ &= (M(x) \pitchfork A(x,-)) * M \\ &= \bigwedge_{y\in A}(M(y) + [M(x), A(x,y)]). \end{aligned}$$

So for any $\delta > 0$, there is a y such that $M(y) + [M(x), A(x,y)] \leq \delta$. This y satisfies $M(y) \leq \delta$, and $A(x,y) \leq M(x) + \delta$. Now given any $\alpha > 0$, one may find a y as required by considering $\delta = min\{\alpha, \epsilon - M(x)\}$. ∎

According to 3.21 and 3.22 above, for any $\mathcal{P}_1$-flat module M, $M = M^- \circ \mathcal{F}(M)$, so that the filter $\mathcal{F}(M)$ is equal to $\mathcal{F} \circ M^- \circ \mathcal{F}(M)$, hence is of type 1 by 3.19. This also shows that the functor $M^- : Fil_1(A) \to \mathcal{P}_1^+(A)$ is essentially surjective on objects, hence is part of an equivalence of category, with equivalence inverse given by $M \mapsto \mathcal{F}(M)$.

To conclude the proof of Theorem 3.14, it remains to reformulate the distance on $Fil_1(A)$ in purely metric terms. This is done in 3.24 below.

Proposition 3.23 *For any left module M on A and any filter $\mathcal{F}$,*

$$[A^{op}, [0,\infty]](M^-(\mathcal{F}), M) \leq lim^+_{\mathcal{F}} M.$$

If $\mathcal{F}$ has type 1 then the inequality above becomes an equality.

PROOF: To simplify notations, let LHS and RHS denote respectively $\bigvee_{x\in A}[M^-(\mathcal{F})(x), M(x)]$ and $\bigwedge_{f\in\mathcal{F}} \bigvee_{z\in f} M(z)$.

According to the definition of $M^-(\mathcal{F})$, for all $x \in A$, for all $f \in \mathcal{F}$, $M^-(\mathcal{F})(x) \geq \bigwedge_{z\in f} A(x,z)$. So for any $x \in A$, any $f \in \mathcal{F}$ and any $\epsilon > 0$, there exists a $z \in f$ such that $A(x,z) \leq M^-(\mathcal{F})(x) + \epsilon$. For such a z, $M(x) \leq M(z) + A(x,z)$ and $M(x) \leq M(z) + M^-(\mathcal{F})(x) + \epsilon$. So,

$$\begin{array}{ll}
 & \forall x \in A, \forall f \in \mathcal{F}, \forall \epsilon > 0, \exists z \in f,\ [M^-(\mathcal{F})(x), M(x)] \leq M(z) + \epsilon, \\
\text{thus} & \forall x \in A, \forall f \in \mathcal{F}, \forall \epsilon > 0,\ [M^-(\mathcal{F})(x), M(x)] \leq (\bigvee_{z\in f} M(z)) + \epsilon, \\
\text{thus} & \forall f \in \mathcal{F}, \forall \epsilon > 0,\ LHS \leq (\bigvee_{z\in f} M(z)) + \epsilon, \\
\text{thus} & \forall f \in \mathcal{F},\ LHS \leq \bigvee_{z\in f} M(z), \\
\text{thus} & LHS \leq RHS.
\end{array}$$

Suppose now that $\mathcal{F}$ has type 1. Consider $\epsilon > 0$. One may find an $f_\epsilon \in \mathcal{F}$ such that for all $z \in f_\epsilon$, $M^-(\mathcal{F})(z) \leq \epsilon$. So,

$$\begin{array}{ll}
 & \text{for any } z \in f_\epsilon,\ M(z) \leq [M^-(\mathcal{F})(z), M(z)] + \epsilon, \\
\text{thus} & \bigvee_{z\in f_\epsilon} M(z) \leq (\bigvee_{x\in A}[M^-(\mathcal{F})(x), M(x)]) + \epsilon, \\
\text{and} & RHS \leq LHS + \epsilon.
\end{array}$$

∎

Corollary 3.24 *For any filters $\mathcal{F}_1$ and $\mathcal{F}_2$ both of type 1,*

$$[A^{op}, [0,\infty]](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) = lim^+_{x\in\mathcal{F}_1} lim^-_{y\in\mathcal{F}_2} A(x,y).$$

A few remarks are in order. First the underlying ordinary category $Fil_1(A)_0$ of $Fil_1(A)$, is a preorder, defined for all $\mathcal{F}_1$, $\mathcal{F}_2$ by

3.25 $\mathcal{F}_1 \to \mathcal{F}_2 \Leftrightarrow \mathcal{F}_1 \supset \mathcal{F} \circ M^-(\mathcal{F}_2)$.

PROOF:

$$\begin{array}{ll}
 & \mathcal{F}_1 \to \mathcal{F}_2 \\
\text{if and only if} & 0 \geq [A^{op}, \mathcal{V}](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) \\
\text{if and only if} & lim^+_{x\in\mathcal{F}_1} M^-(\mathcal{F}_2)(x) = 0 \\
\text{if and only if} & \forall \epsilon > 0, \exists f \in \mathcal{F}_1, \forall x \in f, M^-(\mathcal{F}_2)(x) \leq \epsilon \\
\text{if and only if} & \forall \epsilon > 0, \exists f \in \mathcal{F}_1, f \subset \Gamma(M^-(\mathcal{F}_2))(\epsilon) \\
\text{if and only if} & \mathcal{F}_1 \supset \mathcal{F} \circ M^-(\mathcal{F}_2).
\end{array}$$

∎

Note also that for any filters $\mathcal{F}_1$ and $\mathcal{F}_2$, if $\mathcal{F}_1 \supset \mathcal{F}_2$ then $M^-(\mathcal{F}_1) \Rightarrow M^-(\mathcal{F}_2)$ and for any modules M, N, if $M \Rightarrow N$ then $\mathcal{F}(M) \supset \mathcal{F}(N)$. So that 3.19, 3.21 and 3.22 above shows the existence of an ordinary reflection of the category of filters of type 1 on A with reverse inclusion ordering, in the ordinary category of $\mathcal{P}_1$-flat modules on A. The ordinary category $Fil_1(A)_0$ is exactly the category of fractions deduced from this reflection (see [5], prop.5.3.1).

The following point results also from 3.19. The class of filters of type 1 is the largest class of filters such that fitted with the reverse inclusion ordering, it forms a category with a full reflection in the ordinary category of modules on A, with functors given by the pair of maps $M \mapsto \mathcal{F}(M)$ and $\mathcal{F} \mapsto M^-(\mathcal{F})$.

Observe eventually that 3.15 and 3.23 express respectively the colimits and limits weighted by $\mathcal{P}_1$-flat modules M in terms of the corresponding filters $\mathcal{F}(M)$.

3.2 The theory of filters of type1

Using the categorical machinery of section 2, we explore now the topology induced by the filters of type 1. In particular we shall explicit in topological/metric terms the free $\mathcal{P}_1^+$-cocompletions.

One has a notion of non-symmetric convergence in A. The *neighborhood filter* of $x \in A$, denoted $V_A(x)$, is the filter generated by the family of subsets $\{y \mid A(y,x) \leq \epsilon\}$ with $\epsilon > 0$. Which is to say that $V_A(x)$ is $\mathcal{F}(A(-,x))$. Given a filter $\mathcal{F}$ on A and $x \in A$, we say that $\mathcal{F}$ *converges* to x, that we write $\mathcal{F} \to x$, if and only if $\mathcal{F} \supset V_A(x)$. If $\mathcal{F}$ has type 1 then $\mathcal{F}$ converges to x if and only if $M^-(\mathcal{F}) \Rightarrow A(-,x)$. By Yoneda, this is also equivalent to say that for any $a \in A$,

$$A(x,a) \geq [A^{op},[0,\infty]](M^-(\mathcal{F}), A(-,a)),$$

or according to 3.23 that

$$A(x,a) \geq lim^+_{\mathcal{F}} A(-,a).$$

Definition 3.26 *A filter $\mathcal{F}$ on A has* representative x_0 *if and only if for all $a \in A$,*

$$A(x_0,a) = lim^+_{\mathcal{F}} A(-,a).$$

Which is exactly to say that x_0 is the colimit $M^-(\mathcal{F}) * 1$. In particular if a representative of $\mathcal{F}$ exists then it is unique up to isomorphism. In this case we denote it $rep(\mathcal{F})$. Note that according to Yoneda, $rep(\mathcal{F})$, when it exists, is necessarily the greatest lower bound in A_0 of the set of objects that $\mathcal{F}$ converges to.

Given a filter $\mathcal{F}$ on A and a map $G : A \to B$ the direct image of $\mathcal{F}$ denoted $G(\mathcal{F})$ is the filter on B generated by the family of subsets $G(f)$ for $f \in \mathcal{F}$. It is easy to check for $\mathcal{F}$ and G as above that if G is non-expansive and $\mathcal{F}$ has type 1 then $G(\mathcal{F})$ has again type 1. Moreover,

Proposition 3.27 *Given a filter $\mathcal{F}$ of type 1 on A, and a functor $G : A \to B$, $M^-(G(\mathcal{F})) : B^{op} \to [0,\infty]$ is the (pointwise) left Kan extension of $M^-(\mathcal{F}) : A^{op} \to [0,\infty]$ along G^{op}.*

PROOF: One has the pointwise computation (see [12], (4.17), p.115),

$$\begin{aligned} Lan_{G^{op}}(M^-(\mathcal{F}))(b) &= B(b, G-) * M^-(\mathcal{F}) \\ &= \bigvee_{f \in \mathcal{F}} \bigwedge_{x \in f} B(b, Gx), \text{ according to 3.15,} \\ &= \bigvee_{g \in G(\mathcal{F})} \bigwedge_{y \in g} B(b,y) \\ &= M^-(G(\mathcal{F}))(b). \end{aligned}$$

∎

Note that we could already infer from 2.10 that for any filter $\mathcal{F}$ and any functor $G : A \to B$, if $M^-(\mathcal{F})$ is $\mathcal{P}_1$-flat then $Lan_{G^{op}}(M^-(\mathcal{F}))$ is also $\mathcal{P}_1$-flat. Now remark that for any weight $F : A^{op} \to \mathcal{V}$, for any $G : A \to B$ and any $H : B \to C$ with B and C small, $Lan_{G^{op}}(F) * H \cong F * HG$. So that one has the following consequence of 3.27.

3.28 *Given a filter $\mathcal{F}$ of type 1 on A and a non expansive map $G : A \to B$, $M^-(\mathcal{F}) * G$ is (up to isomorphism) the representative of the filter $G(\mathcal{F})$ of type 1.*

We shall call a general metric space A *(type 1)-complete*, when any filter of type 1 on A admits a representative, according to 3.28, this is to say when the associated category is $\mathcal{P}_1^+$*-cocomplete*. Now consider a filter $\mathcal{F}$ of type 1 on K, and non-expansive maps $G : K \to A$, and a functor $H : A \to B$. Then H (as a functor) preserves the colimit $M^-(\mathcal{F}) * G$ if and only H (as a non-expansive map) preserves the representative of $G(\mathcal{F})$, i.e.

$$H(rep(G(\mathcal{F}))) = rep(H \circ G(\mathcal{F})).$$

To sum up: the $[0, \infty]$-functors preserving the $\mathcal{P}_1^+$-colimits are exactly the non-expansive maps preserving the representatives of filters of type 1. Now a translation of Kelly's theorem (2.1) gives the following completion for metric spaces.

Theorem 3.29 *For any general metric space A, there exists a (type 1)-complete metric space $\bar{A}$ together with a non expansive map $i_A : A \to \bar{A}$, such that for any (type 1)-complete general metric space B, composing with i_A defines an equivalence of general metric space:*

$$- \circ i_A : [\bar{A}, B]' \cong [A, B]$$

where $[\bar{A}, B]'$ denotes the sub-metric space of $[\bar{A}, B]$ of maps preserving the representatives of filters of type 1.
Any $\bar{A}$ as above is equivalent to the metric space $Fil_1(A)$, and the embedding i_A is, up to this equivalence, the non-expansive map $A \to Fil_1(A)$ sending a point x to its neighborhood filter $V_A(x)$.

Let us call therefore $Fil_1(A)$ the *completion of type 1* of A. Let us add that with A and B as above, for any $f : A \to B$, the unique extension $\bar{f} : Fil_1(A) \to B$ of f through i_A sends any filter $\mathcal{F}$ to the representative of its direct image by f in B. To check this, just come back to the categorical formulation. From [12] Theorem 4.97, $\bar{f}$ is the left Kan extension of f along i_A and sends any M in $\mathcal{P}_1^+(A)$ to $M * f$. Translate then using 3.28.

Let us mention the following result that can be inferred from purely categorical arguments. The limits in 3.24 "commute" when the first argument is Cauchy:

3.30 *For any Cauchy filter $\mathcal{F}_1$ and any filter $\mathcal{F}_2$ of type 1,*

$$[A^{op}, [0, \infty]](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) = lim^-_{y \in \mathcal{F}_2} lim^+_{x \in \mathcal{F}_1} A(x, y).$$

To see this we shall need the following result see for instance [3]-remark 4 p.171 or [14] section 6.

Proposition 3.31 *Given a complete and cocomplete monoidal closed $\mathcal{V}$, if a $\mathcal{V}$-module $M : I \multimap C$ has a right adjoint $\tilde{M} : C \multimap I$ then for any $\mathcal{V}$-module $N : I \multimap C$, $[C^{op}, \mathcal{V}](M, N) \cong \tilde{M} * N$.*

We can now prove 3.30.

PROOF: Recall that for any Cauchy filter $\mathcal{F}$ on A, $M^-(\mathcal{F}) = M^+(\mathcal{F}) = M^l(\mathcal{F})$ and this left module on A has right adjoint the module $M^r(\mathcal{F})$ defined by the map $x \mapsto lim^+_{y\in\mathcal{F}} A(y,x) = lim^-_{y\in f} A(y,x)$. So according to 3.31 and 3.15, for any Cauchy filter $\mathcal{F}_1$ and any filter $\mathcal{F}_2$ of type 1,

$$\begin{aligned}[A^{op}, [0,\infty]](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) &= M^r(\mathcal{F}_1) * M^-(\mathcal{F}_2) \\ &= lim^-_{y\in\mathcal{F}_2} M^r(\mathcal{F}_1)(y) \\ &= lim^-_{y\in\mathcal{F}_2} lim^+_{x\in\mathcal{F}_1} A(x,y).\end{aligned}$$

∎

3.3 $\mathcal{P}_\aleph$-flat modules and filters of type $\aleph$.

$\aleph$ will denote in this section any infinite regular cardinal. We turn now to the case of $\mathcal{P}_\aleph$-flat modules and their corresponding filters, the so-called filters of type $\aleph$.

Definition 3.32 *A filter $\mathcal{F}$ on A has* type $\aleph$ *if and only if for any $\epsilon > 0$, there exists an $f \in \mathcal{F}$ such that for any family of elements $(x_i)_{i\in I}$ of f, with $\sharp I < \aleph$, for any $g \in \mathcal{F}$, there exists $y \in g$ such that $A(x_i, y) \leq \epsilon$.*

Note the inclusion of classes of filters:

3.33 *Cauchy $\Rightarrow$ type $\aleph$ $\Rightarrow$ type ω $\Rightarrow$ type 1.*

Also when A is symmetric, that is when $A(x,y) = A(y,x)$, filters of type ω are also Cauchy. (We shall see later a few consequences of this fact.)

With the above definition of filters, the equivalence of Theorem 3.14 restricts to the full subcategories of $\mathcal{P}_\aleph$-flat presheaves and filters of type $\aleph$.

Theorem 3.34 *The full subcategory of $Fil_1(A)$ induced by the filters of type $\aleph$ is equivalent to $\mathcal{P}^+_\aleph(A)$, the equivalence functors being given by the maps $\mathcal{F} \mapsto M^-(\mathcal{F})$ and $M \mapsto \mathcal{F}(M)$.*

This results from Theorem 3.14 and 3.35 and 3.37 below.

3.35 *For any $\mathcal{P}_\aleph$-flat module $M : I \multimap A$, $\mathcal{F}(M)$ has type $\aleph$.*

PROOF: If $- * M$ preserves conical $\aleph$-limits then it preserves in particular the terminal object and according to 3.20, $\mathcal{F}(M)$ is a filter on A. The fact that the filter basis $\Gamma(M)$ generates a filter of type $\aleph$ is a consequence of the following result. ∎

3.36 *If $M : I \multimap A$ is $\mathcal{P}_{\aleph}$-flat then for any $\epsilon > 0$ and any family $(x_i)_{i\in I}$, with $\sharp I < \aleph$, such that for all i, $M(x_i) \leq \epsilon/2$ and any $\alpha > 0$, there is a y such that $M(y) \leq \alpha$ and for all $i \in I$, $A(x_i, y) \leq \epsilon$.*

PROOF: $- * M$ preserves conical $\aleph$-limits and cotensors. Consider $\epsilon > 0$ and a family $(x_i)_{i\in I}$'s with $\sharp I < \aleph$ such that $M(x_i) \leq \epsilon/2$. Let us write $\epsilon' = \bigvee_{i\in I} M(x_i)$. Then

$$\begin{aligned}
0 &= [\epsilon', \textstyle\bigvee_{i\in I} M(x_i)] \\
&= [\epsilon', \textstyle\bigvee_{i\in I}(A(x_i, -) * M)] \\
&= [\epsilon', (\textstyle\bigvee_{i\in I} A(x_i, -)) * M] \\
&= (\epsilon' \pitchfork (\textstyle\bigvee_{i\in I} A(x_i, -)) * M \\
&= \textstyle\bigwedge_{y\in A}(M(y) + [\epsilon', \bigvee_{i\in I} A(x_i, y)]).
\end{aligned}$$

So for any $\delta > 0$, there is a y such that $M(y) + [\epsilon', \bigvee_{i\in I} A(x_i, y)] \leq \delta$. This y satisfies $M(y) \leq \delta$, and for all i, $A(x_i, y) \leq \epsilon' + \delta$. Now given any $\alpha > 0$, one may find a y as required by considering $\delta = min\{\alpha, \epsilon - \epsilon'\}$. ∎

3.37 *If the filter $\mathcal{F}$ on A has type $\aleph$ then $- * M^-(\mathcal{F})$ preserves conical $\aleph$-limits, i.e. for any family $(N_i)_{i\in I}$ of right modules on A, with $\sharp I < \aleph$,*

$$\bigwedge_{x\in A}(M^-(\mathcal{F})(x) + \bigvee_{i\in I} N_i(x)) = \bigvee_{i\in I}\bigwedge_{x\in A}(M^-(\mathcal{F})(x) + N_i(x)).$$

PROOF: We only need to prove

$$\bigwedge_{x\in A}(M^-(\mathcal{F})(x) + \bigvee_{i\in I} N_i(x)) \leq \bigvee_{i\in I}\bigwedge_{x\in A}(M^-(\mathcal{F})(x) + N_i(x)).$$

Let $\epsilon > 0$. If there is a filter $\mathcal{F}$ on A then A is not empty and for each $i \in I$, there is an $x_i \in A$ such that

$$N_i * M^-(\mathcal{F}) + \epsilon = \bigwedge_{x\in A}(M^-(\mathcal{F})(x) + N_i(x)) + \epsilon \geq M^-(\mathcal{F})(x_i) + N_i(x_i).$$

Let $f \in \mathcal{F}$. Given a family of x_i's as above, for each i, $M^-(\mathcal{F})(x_i) \geq \bigwedge_{y\in f} A(x_i, y)$, thus there is an $y_i \in f$ such that $M^-(\mathcal{F})(x_i) + \epsilon \geq A(x_i, y_i)$ and

$$\begin{aligned}
2\cdot\epsilon + N_i * M^-(\mathcal{F}) &\geq A(x_i, y_i) + N_i(x_i) \\
&\geq N_i(y_i).
\end{aligned}$$

Since $\mathcal{F}$ has type $\aleph$, we can choose f so that for the $y_i \in f$ as above, for all $g \in \mathcal{F}$, there exists $z \in g$ such that for all i, $A(y_i, z) \leq \epsilon$.

Thus for all $\epsilon > 0$, for all $g \in \mathcal{F}$, there exists $z \in g$ such that for all $i \in I$,

$$\begin{aligned}
3\cdot\epsilon + N_i * M^-(\mathcal{F}) &\geq A(y_i, z) + N_i(y_i) \text{ for some suitable } y_i\text{'s}, \\
&\geq N_i(z).
\end{aligned}$$

and thus

$$\begin{array}{ll} & \forall\epsilon>0, \forall g\in\mathcal{F}, \exists z\in g,\ \epsilon+\bigvee_{i\in I}(N_i * M^-(\mathcal{F})) \geq \bigvee_{i\in I} N_i(z); \\ \text{thus} & \forall\epsilon>0, \forall g\in\mathcal{F},\ \epsilon+\bigvee_{i\in I}(N_i * M^-(\mathcal{F})) \geq \bigwedge_{z\in g}\bigvee_{i\in I} N_i(z); \\ \text{thus} & \forall g\in\mathcal{F}, \bigvee_{i\in I}(N_i * M^-(\mathcal{F})) \geq \bigwedge_{z\in g}\bigvee_{i\in I} N_i(z). \end{array}$$

So

$$\begin{aligned} \bigvee_{i\in I}(N_i * M^-(\mathcal{F})) &\geq \bigvee_{g\in\mathcal{F}}\bigwedge_{z\in g}\bigvee_{i\in I} N_i(z) \\ &= (\bigvee_{i\in I} N_i) * M^-(\mathcal{F}), \text{ according to 3.15.} \end{aligned}$$

■

One obtains also (type $\aleph$)-completions for general metric spaces. Let $Fil_\aleph(A)$ stand for the full subcategories of $Fil_1(A)$ generated by the filters of type $\aleph$. Then Theorem 3.29 still holds after that "type $\aleph$" and "$Fil_\aleph(A)$" has been substituted everywhere respectively to "type 1" and "$Fil_1(A)$". Actually the only point to check to establish this, is that the direct image by a non expansive map of a filter a type $\aleph$, is again of type $\aleph$; which is straightforward.

3.4 Examples

We give now examples of filters of types 1 and ω, complete spaces and completions.

Recall from [12] (3.74) that any monoidal closed $\mathcal{V}$ that is complete and cocomplete as an ordinary category, is complete and cocomplete as a $\mathcal{V}$-category. Thus

3.38 $[0,\infty]$ *is (type* 1*)-complete.*

Since $\mathcal{P}_1 \supset \mathcal{C}$ one has $\mathcal{P}_1^+ \subset \mathcal{C}^+$ and thus according to 2.11, one has also the following.

Proposition 3.39 *The (type* 1*)-completion of* $[0,\infty]$ *is equivalent to* $[0,\infty]$.

It might be worth detailing a bit the situation here. For a filter $\mathcal{F}$ on $[0,\infty]$, we write $\liminf(\mathcal{F})$ for $lim^-_{\mathcal{F}}id$ which is $\bigvee_{f\in\mathcal{F}}\bigwedge_{x\in f} x$. If $\mathcal{F}$ has type 1 then $\liminf(\mathcal{F})$ is just $M^-(\mathcal{F}) * 1$ by Proposition 3.15, and according to Proposition 2.11, $M^-(\mathcal{F}) : \mathcal{V}^{op} \to \mathcal{V}$ is isomorphic to $[-, \liminf(\mathcal{F})]$. In the latter case, $\mathcal{F} \supset V_{\mathcal{V}}(\liminf \mathcal{F})$ i.e. $\mathcal{F}$ converges to $\liminf(\mathcal{F})$.

Proposition 3.40 *Filters of type* 1 *on* $[0,\infty]$ *have type* ω.

PROOF: Let $\mathcal{F}$ be a filter of type 1 on $[0,\infty]$. Consider $\epsilon > 0$ then there exists an $f \in \mathcal{F}$ such that for any $x \in f$, for any $g \in \mathcal{F}$, there exists a y such that $A(x,y) \leq \epsilon$. For this f, for any finite family of elements x_i in $\mathcal{F}$ and for any $g \in \mathcal{F}$, one may find elements $y_i \in g$, such that for all i, $[x_i, y_i] \leq \epsilon$, i.e. $y_i \leq x_i + \epsilon$. Choosing the least of those y_i's, say z, one has $[x_i, z] \leq \epsilon$ for all i. ■

We investigate now the case of *symmetric* spaces.

3.41 *If A is symmetric,*

- (1) *filters of type ω on A are Cauchy;*
- (2) *For any Cauchy filter $\mathcal{F}$, $M^l(\mathcal{F})$ and $M^r(\mathcal{F})$ have the same underlying map.*
- (3) *Any left adjoint module on A has the same underlying map as its right adjoint;*
- (4) *For any left adjoint module M on A, $\mathcal{F}(M) = \mathcal{F}^s(M)$;*
- (5) *$\mathcal{P}_\omega$-flat modules are left adjoint;*
- (6) *The ordinary category $\mathcal{Q}(A)_0$ is discrete.*

PROOF: (1) – that was already mentioned – and (2) are trivial. (3) holds since for any left module M with right adjoint $\tilde{M}$, according to (2) their underlying maps satisfy $M = M^l \circ \mathcal{F}^s(M) = M^r \circ \mathcal{F}^s(M) = \tilde{M}$. (4) is straightforward from (3). To prove (5), consider the successive equivalences

> M is $\mathcal{P}_\omega$-flat
> if and only if $M = M^-(\mathcal{F})$ for a filter of type ω
> if and only if $M = M^-(\mathcal{F})$ for a Cauchy filter (according to (1)),
> if and only if M is left adjoint.

Now we show (6), namely: the underlying subcategory of the full subcategory of presheaves $[A^{op}, [0,\infty]]$ with objects left adjoint modules is discrete (in the particular case $\mathcal{V} = [0,\infty]$). For any left adjoint module M on A, M has the same underlying map as its right adjoint $\tilde{M}$. Now consider another left adjoint module N on A, with right adjoint $\tilde{N}$. Then $M \Rightarrow N$ if and only if $\forall x \in A, M(x) \geq N(x)$ if and only if $\forall x \in A, \tilde{M}(x) \geq \tilde{N}(x)$ if and only if $\tilde{M} \Rightarrow \tilde{N}$. But also if $M \Rightarrow N$ then $1 \Rightarrow \tilde{M}N$ since $M \dashv \tilde{M}$ and then $\tilde{N} \Rightarrow \tilde{M}$ since $N \dashv \tilde{N}$. So $M \Rightarrow N$ if and only if $M = N$. ∎

The assertion (1) above tells us that when the general metric space A is symmetric, it happens that $\mathcal{P}^+_\omega[A] = \mathcal{P}^+[A]$, which is also to say that the completion of type ω of A is its Cauchy completion. Nevertheless even when A is symmetric, its completion of type 1 may be not symmetric. We show below that it consists of the set of non-empty closed subsets of A with what one could call a "semi-Hausdorff" distance.

Proposition 3.42 *The completion of type* 1 *of a symmetric A is the set of non-empty closed subsets of its Cauchy-completion $\bar{A}$ with pseudo distance d given by $d(X,Y) = \bigvee_{x\in X} \bigwedge_{y \in Y} \bar{A}(x,y)$.*

To prove this result, we shall establish a characterization of filters of type 1 as certain colimits of *forward Cauchy sequences*. These sequences belong to folklore and were introduced as a generalization of the classical Cauchy sequences for

non-symmetric metric spaces. It is not known by the author whether they admit a reasonable description in categorical terms. Nevertheless these sequences define peculiar filters of type ω and as such, their whole class has a surprising density property in $Fil_1(A)_0$ (see Theorem 3.43 below).

Given a sequence $(x_n)_{n\in\mathbb{N}}$ on A, the associated filter, still denoted (x_n), has basis the family of sets $\{x_p \mid p \geq n\}$. We say that (x_n) is:
- *of type 1*, respectively *of type* ω, if the associated filter is so;
- *forward Cauchy* if and only if $\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall m \geq n \geq N, A(x_n, x_m) \leq \epsilon$.
Note that any forward Cauchy sequence is obviously of type ω.

Theorem 3.43 *$Fil_1(A)_0$ has all the colimits of non-empty diagrams and filters of type 1 are colimits in $Fil_1(A)_0$ of non-empty diagrams with values forward Cauchy sequences.*

This result will follow from 3.46 and 3.47 below. We note first that the base $[0,\infty]$ has a very peculiar property:

3.44 *The conical terminal object commutes with conical colimits of non-empty diagrams in $[0,\infty]$.*

PROOF: The weight for the conical terminal object is the unique functor $! : \emptyset \to [0,\infty]$, where $\emptyset$ denotes the empty category. Also the functor category $[\emptyset, [0,\infty]]$ is isomorphic to the terminal category 1 with one object $*$ and hom $1(*,*) = 0$. Therefore the fact that for any $v \in [0,\infty]$, $[v, 0] = 0$ shows that the limit $\{!, !\}$ is 0. Now we consider an ordinary category J and let $J^\sharp$ denote the free $[0,\infty]$-category over J. Since limits in functor categories are pointwise, the limit weighted by $!$ of the unique functor $\emptyset \to [J^\sharp, [0,\infty]]$, denoted again $\{!, !\}$, is the constant functor $J^\sharp \to [0,\infty]$ to 0. The conical colimit of this functor is 0 since J is non-empty. On the other hand, the unique functor $J^\sharp \to [\emptyset, [0,\infty]] \cong 1$ is necessarily the constant one with image the unique $! : \emptyset \to [0,\infty]$, its conical colimit is necessarily the functor $! : \emptyset \to [0,\infty]$ which limit weighted by $! : \emptyset \to [0,\infty]$ is $\{!, !\}$, which is known to be 0. ∎

According to this and the fact that conical colimits of non-empty diagrams commute also in $[0,\infty]$ with cotensors (3.17), one has:

3.45 *the class of weights for conical colimits of non-empty diagrams is contained in $\mathcal{P}_1^+$.*

Therefore, since $\mathcal{P}_1^+$ is saturated, one has the following result:

3.46 *$\mathcal{P}_1^+(A)$ is closed in $[A^{op}, [0,\infty]]$ under the formation of conical colimits of non empty diagrams.*

3.47 *Given a filter $\mathcal{F}$ of type 1 on A and a left module M such that $M^-(\mathcal{F}) \not\to M$ (i.e. $M^-(\mathcal{F}) \not\geq M$), there is a forward Cauchy sequence (y_n) such that $(y_n) \to \mathcal{F}$ and $M^-(y_n) \not\to M$.*

PROOF: The arguments will be technical and we introduce some convenient notation. For any $f \subset A$, any $\epsilon > 0$ and, any $F \subset \wp\wp(A)$, we let $P(f, \epsilon, F)$ denote the property:
"for all x in f, for all $g \in F$ there exists $y \in g$ such that $A(x, y) \leq \epsilon$".

Now, by hypothesis there exists $x \in A$ such that $\bigvee_{f \in \mathcal{F}} \bigwedge_{y \in f} A(x, y) < M(x)$. Consider such an x. There exists $\alpha > 0$ such that for any $f \in \mathcal{F}$, there exists $y \in f$ such that $A(x, y) + \alpha < M(x)$. Note then that for such a y, $A(x, y)$ is necessarily finite.

Since $\mathcal{F}$ has type 1, one can define a sequence (f_n) of elements of $\mathcal{F}$ such that for all $n \in I\!N$, $f_{n+1} \subset f_n$ and $P(f_n, \alpha \cdot 2^{-2-n}, \mathcal{F})$. (f_n) is defined inductively as follows.

Choose first f_0 such that $P(f_0, \alpha \cdot 2^{-2}, \mathcal{F})$.

If f_n is defined then one can find $g \in \mathcal{F}$ such that $P(g, \alpha \cdot 2^{-2-(n+1)}, \mathcal{F})$ and let $f_{n+1} = f_n \cap g$.

Then one can build a sequence (y_n) where for all integer n, $y_n \in f_n$, y_0 is such that $A(x, y_0) + \alpha < M(x)$, and for all integer n, $y_n \in f_n$, $A(y_n, y_{n+1}) \leq \alpha \cdot 2^{-2-n}$.

Actually this ensures that:
(1) (y_n) is forward Cauchy;
(2) $(y_n) \to \mathcal{F}$;
(3) $M^-(y_n) \not\Rightarrow M$.

(1) holds since for all $n \leq p \in I\!N$,

$$\begin{aligned} A(y_n, y_p) &\leq A(y_n, y_{n+1}) + ... + A(y_{p-1}, y_p) \\ &\leq \alpha \cdot (2^{-2-n} + 2^{-2-(n+1)} + ...) \\ &= \alpha \cdot 2^{-1-n}. \end{aligned}$$

(2) holds since (y_n) is forward Cauchy, for any $n \in I\!N$, $\{y_p / p \geq n\} \subset f_n$ and $P(f_n, \alpha \cdot 2^{-2-n}, \mathcal{F})$.

(3) holds since for all $n \in I\!N$,

$$\begin{aligned} A(x, y_n) &\leq A(x, y_0) + A(y_0, y_1) + ... + A(y_{n-1}, y_n) \\ &\leq A(x, y_o) + \alpha/2 \end{aligned}$$

so $A(x, y_n) + \alpha/2 < M(x)$. Thus $M^-(y_n)(x) = \bigvee_{n \in I\!N} \bigwedge_{p \geq n} A(x, y_p) < M(x)$. ∎

To finish proving Theorem 3.43, it remains to see that any filter $\mathcal{F}$ of type 1 dominates at least one forward Cauchy sequence. But this holds for such an $\mathcal{F}$ according to 3.47, since $lim^+_{\mathcal{F}} M^-(\mathcal{F}) = 0$ and thus $M^-(\mathcal{F}) \not\Rightarrow +\infty$ where $+\infty$

denotes here the constant module with value $+\infty$.

We come back now to the proof of Proposition 3.42.

PROOF: Since A is symmetric, its Cauchy completion is equivalent to the metric space, say B, with objects Cauchy filters on A with "symmetric" distance d given for all φ, ψ by $d(\varphi,\psi) = [A^{op},[0,\infty]](M^-(\varphi),M^-(\psi)) = M^r(\varphi) * M^-(\psi)$ by Proposition 3.31. This metric space is also symmetric (according to 3.41) and forward Cauchy sequences in B are just the Cauchy ones.

Consider a filter $\mathcal{F}$ of type 1 on A and let $\bar{\mathcal{F}}$ denote the set of Cauchy filters ϕ such that $\phi \to \mathcal{F}$. According to Theorem 3.43 and 3.46, $\bar{\mathcal{F}}$ is not empty and $M^-(\mathcal{F})$ is the pointwise conical colimit in $[A^{op},\mathcal{V}]$:

$$M^-(\mathcal{F}) = \bigwedge_{\varphi\in\bar{\mathcal{F}}} M^-(\varphi).$$

Now for any subset X of B that satisfies the property

$$(*)\ \ M^-(\mathcal{F}) = \bigwedge_{\varphi\in X} M^-(\varphi),$$

and any Cauchy filter ψ on A, one has:

$$\begin{aligned} d(\psi,\mathcal{F}) &= [A^{op},[0,\infty]](M^-(\psi),M^-(\mathcal{F})) \\ &= [A^{op},[0,\infty]](M^-(\psi),\textstyle\bigwedge_{\varphi\in X} M^-(\varphi)) \\ &= \textstyle\bigwedge_{\varphi\in X}[A^{op},[0,\infty]](M^-(\psi),M^-(\varphi))\ (**) \\ &= \textstyle\bigwedge_{\varphi\in X} d(\psi,\varphi). \end{aligned}$$

where $(**)$ above holds since the presheaf $M^-(\psi) : A^{op} \to [0,\infty]$ is a small projective (or equivalently the module $M^-(\psi) : I \longrightarrow\!\!\!\!\!\circ\ A$ is left adjoint).

As a consequence of this, one has that for any subset X of B satisfying $(*)$, the adherence $\bar{X}$ of X in B is $\bar{\mathcal{F}}$. Therefore $\bar{\mathcal{F}}$ is the only closed subset X in B satisfying the condition $(*)$. This result, together with Theorems 3.14 and 3.43, show that the equation $(*)$ defines a bijection between $\mathcal{P}_1$-flat modules on A and and non-empty subsets of B.

Eventually given two filters on A of type 1, $\mathcal{F}_1$ and $\mathcal{F}_2$,

$$\begin{aligned} [A^{op},[0,\infty]](M^-(\mathcal{F}_1),M^-(\mathcal{F}_2)) &= [A^{op},[0,\infty]](\textstyle\bigwedge_{\varphi\in\bar{\mathcal{F}}_1} M^-(\varphi),M^-(\mathcal{F}_2)) \\ &= \textstyle\bigvee_{\varphi\in\bar{\mathcal{F}}_1}[A^{op},[0,\infty]](M^-(\varphi),M^-(\mathcal{F}_2)) \\ &= \textstyle\bigvee_{\varphi\in\bar{\mathcal{F}}_1}[A^{op},[0,\infty]](M^-(\varphi),\bigwedge_{\psi\in\bar{\mathcal{F}}_2} M^-(\psi)) \\ &= \textstyle\bigvee_{\varphi\in\bar{\mathcal{F}}_1}\bigwedge_{\psi\in\bar{\mathcal{F}}_2} d(\varphi,\psi), \end{aligned}$$

since, again, $M^-(\varphi)$ is a small projective for any Cauchy filter φ. ∎

4 The case $\mathcal{V} = \mathbf{2}$.

Preorders as enrichments over the category $\mathbf{2}$ were mentioned in [16]. After a brief reminder, we shall characterize in the context $\mathcal{V} = \mathbf{2}$ the $\mathcal{P}_1$- and $\mathcal{P}_\aleph$-

flatnesses and the associated free cocompletions. We shall see that the free $\mathcal{P}_\omega^+$-cocompletion is the classic *dcpo* completion.

2 stands for the two-object category generated by the graph $0 \longrightarrow 1$. It is a partial order and has a monoidal structure with tensor $\wedge$ (the logical "and") and unit 1. **2** is closed since for all $x, y, z \in \mathbf{2}$,

$$x \wedge y \leq z \Leftrightarrow x \leq (y \Rightarrow z)$$

where $\Rightarrow$ denotes the usual entailment relation. Small **2**-categories are just preorders: for any small **2**-category A, its associated preorder is defined by $x \to y$ if and only if $A(x, y) = 1$. Along the same line there is a bijection between **2**-functors and monotonous maps. Any **2**-module $M : I \multimap A$ corresponds to a downset $\mathcal{I}_M = \{x \mid M(x) = 1\}$ on the preorder A, and this correspondence between modules and downsets is bijective. Also for any preorders A and B, the set of monotonous maps is considered as pointwise ordered, which corresponds to the **2**-enriched categorical structure of the functor category $[A, B]$. Recall that small **2**-categories are always Cauchy-complete. Also the *downward completion* of a preorder may be described as its set of downsets ordered by inclusion, and is its free completion. Via the above translations, the downward completion is just the free cocompletion of **2**-categories.

Let us turn now to the $\mathcal{P}_1$- and $\mathcal{P}_\aleph$-flatnesses. In the rest of this section, A denotes a small **2**-category that we freely consider as a preorder, and $\aleph$ is any infinite regular cardinal. Using 3.3, 3.4 and 3.5 again, one gets the following.

4.1 *For any module* $M : I \multimap A$ *,*

- $- * M : [A, \mathbf{2}] \to \mathbf{2}$ *preserves the (conical) terminal object i.e.* $1 * M = 1$, *if and only if*

$$(1) \quad \bigvee_{x \in A} M(x) = 1;$$

- $- * M$ *preserves conical* $\aleph$*-limits if and only if*
(2) *For any family of right modules* $N_i : A \multimap I$ *,* i *ranging in* I *and* $\sharp I < \aleph$*,*

$$\bigvee_{x \in A} (M(x) \wedge \bigwedge_{i \in I} N_i(x)) = \bigwedge_{i \in I} (\bigvee_{x \in A} M(x) \wedge N_i(x));$$

- $- * M$ *preserves cotensors if and only if*
(3) *For any* $v \in \mathbf{2}$ *and any right module* $N : A \multimap I$ *,*

$$\bigvee_{x \in A} (M(x) \wedge (v \Rightarrow N(x))) \quad = \quad (v \Rightarrow \bigvee_{x \in A} (M(x) \wedge N(x))).$$

Condition (1) above is equivalent to the fact that $\mathcal{I}_M$ is not empty. Condition (3) reduces for $v = 1$ to the trivial equation $N * M = N * M$. For $v = 0$, it

reduces to $1 = \bigvee_{x \in A} M(x)$, that is (1) again. Recall that a downset $\mathcal{I}$ of A is said $\aleph$-*directed* if and only if it satisfies the property:
$(*)$ Any subset of $\mathcal{I}$ of cardinality strictly less than $\aleph$ has an upper bound in $\mathcal{I}$. Note that such an I is necessarily not empty. Now observe that condition (2) above is equivalent to the fact that $\mathcal{I}_M$ satisfies $(*)$. So one has a bijection between the following on A:

- $\mathcal{P}_1$-flat left modules and non-empty downsets,
- $\mathcal{P}_\aleph$-flat left modules and $\aleph$-directed downsets.

Thus $\mathcal{P}_1^+(A)$ is equivalent as a preorder to the set of non-empty downsets of A with inclusion ordering, whereas $\mathcal{P}_\aleph^+(A)$ is equivalent to the set the $\aleph$-directed downsets of A with inclusion ordering, that we shall write $\aleph$-*dcpo*(A).

Eventually given a weight $M : A^{op} \to \mathbf{2}$ and $G : A \to B$ with B small, $b \in B$ is the colimit $M * G$ if and only if b is the least upper bound in the preorder B of the downset generated by the direct image of $\mathcal{I}_M$ by G. Also a functor $H : B \to C$ preserves $M * G$ as above if and only if the corresponding monotonous map preserves the least upper bound of $G(\mathcal{I}_M)$. Remember that a partial order is said $\aleph$-*directed complete* or, for short, is a $\aleph$-*dcpo*, when it admits all least upper bounds for $\aleph$-directed subsets. $\aleph$-dcpos with maps preserving $\aleph$-directed least upper bounds form the category $\aleph$-*dcpo*. So we know from categorical considerations that $\aleph$-*dcpo*(A) is $\aleph$-directed complete.

From 2.1 again, one can deduce a few completions for preorders, and in particular the well-known following one.

Theorem 4.2 ($\aleph$-dcpo completion) *Given a preorder A, $\aleph$-dcpo(A) and the order preserving map $i_A : A \to \aleph\text{-}dcpo(A)$ sending a to the downset generated by a, satisfy the following universal property. Composing with i_A defines an equivalence of preorders:*

$$- \circ i_A : [\aleph\text{-}dcpo(A), B]' \cong [A, B]$$

where:

- $[A, B]$ *is the pointwise preorder of monotonous maps $A \to B$;*
- $[\aleph\text{-}dcpo(A), B]'$ *is the pointwise preorder of maps $\aleph\text{-}dcpo(A) \to B$ that preserve $\aleph$-directed least upper bounds.*

The case $\aleph = \omega$ is quite popular in Computer Science where our ω-dcpos and ω-dcpo morphisms are called respectively *dcpos* and *continuous maps*.

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